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A Note on the Nonuniqueness for Some Quasilinear Eigenvalue Problem

P. DRÁBEK
Department of Mathematics
University of West Bohemia
P.O. Box 314, 306 14 Pilsen, Czech Republic
pdrabek@kma.zcu.cz

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Abstract—Let $1 < p < N$ and $\Omega_r := \{x \in \mathbb{R}^N : 0 < a < |x| < b < \infty\}$. We prove that there exists q, $p < q < p^* = Np/(N-p)$ such that the extremal for the optimal constant $C := C(p, q, \Omega_r)$ of the embedding $W_0^{1,p}(\Omega_r) \hookrightarrow\hookrightarrow L^q(\Omega_r)$ is not a radial function (and hence, not unique). As a consequence of this fact, we obtain a nonuniqueness result for

$$-\Delta_p u = |u|^{q-2}u, \qquad u > 0 \text{ in } \Omega_r, u = 0 \text{ on } \partial\Omega_r, \|\nabla u\|_p = C^{q/(p-q)}. \tag{*}$$

Keywords—The p-Laplacian, Nonhomogeneous eigenvalue problem, The best Sobolev constant, Nonuniqueness.

1. STATEMENT OF THE RESULT

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $1 < p < N$ and $1 < q < p^* = Np/(N-p)$. Then the following embedding:

$$W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \tag{1}$$

holds and the optimal value of the constant $C := C(p, q, \Omega)$ in the inequality

$$\|u\|_q \le C\|\nabla u\|_p \tag{2}$$

can be characterized by

$$C := \sup\{\|u\|_q : \|\nabla u\|_p = 1\}. \tag{3}$$

Due to the compactness of (1), the sup in (3) is achieved at a certain nonnegative function $u_{pq}(\Omega) \in W_0^{1,p}(\Omega)$ and equality in (2) then holds for any $u_\tau = \tau u_{pq}(\Omega), \tau \in \mathbb{R}$. The Lagrange

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multiplier method yields that u_τ is a weak solution of

$$-\Delta_p u = \tau^{q-p} C^{-q} |u|^{q-2} u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{4}$$

The Moser-type iterations [3] together with the regularity results [4,5] and the Harnack-type inequality [6] imply that $u_\tau \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and $u_\tau > 0$ in Ω.

For $1 < q < p$, it follows from [7] that $u_{pq}(\Omega)$ is unique and the same result follows from [8] if $q = p$. In this note, we want to show that, in general, this is not case if $p < q < p^*$.

THEOREM 1. *Let $\Omega_r := \{x \in \mathbb{R}^N : 0 < a < |x| < b < \infty\}$. There exists q, $p < q < q^*$ such that the function $u_{pq}(\Omega_r)$ is not radially symmetric.*

COROLLARY 1. *Let q be as above. Then the problem (*) does not have unique solution.*

2. THE PROOFS

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $p < q < p^*$. Consider the functional

$$J_q(v) := \frac{1}{p} \int_\Omega |\nabla v|^p - \frac{1}{q} \int_\Omega |v|^q$$

associated to the problem

$$-\Delta_p v = |v|^{q-2} v \text{ in } \Omega, \qquad v = 0 \text{ on } \partial\Omega. \tag{5}$$

It follows from (4) that $u(q) := C^{q/(p-q)} u_{pq}(\Omega)$ is a solution of (5), and hence, also a critical point of J_q. Moreover, we have

$$J_q(u(q)) = \min\left\{ J_q(v) : \|\nabla v\|_p = C^{q/(p-q)} \right\}.$$

Furthermore, the real function

$$t \mapsto J_q(tu(q)) = \left(\frac{t^p}{p} - \frac{t^q}{q}\right) C^{pq/(p-q)} \tag{6}$$

is increasing in $(0,1)$ and decreasing in $(1,\infty)$. Let $\tau_0 > 1$ be so large that $J(v_0) < 0, v_0 = \tau_0 u(q)$. Set

$$\mathcal{C} := \left\{ g \in C\left([0,1]; W_0^{1,p}(\Omega)\right) : g(0) = 0, g(1) = v_0 \right\}$$

and define

$$C_q = \inf_{g \in \mathcal{C}} \max_{t \in [0,1]} J_q(g(t)).$$

LEMMA 1. *The value C_q is achieved in $u(q)$.*

PROOF. Let $g \in \mathcal{C}$ be arbitrary. Then g must intersect $\{u \in W_0^{1,p}(\Omega) : \|\nabla u\|_p = C^{q/(p-q)}\}$, i.e., $C_q \geq J_q(u(q))$. On the other hand, define $g_0 \in \mathcal{C}$ as $g_0(t) = tv_0, t \in [0,1]$. The properties of (6) imply

$$\max_{t \in [0,1]} J_q(g_0(t)) = J_q\left(g_0\left(\frac{1}{\tau_0}\right)\right) = J_q(u(q)), \text{ i.e., } C_q \leq J_q(u(q)). \qquad \blacksquare$$

Now it follows from Proposition 1, Lemma 5, and Theorem 6 in [9] that there exists a sequence $q_k \nearrow p^*$ such that

$$(u(q_k))^{p^*} \rightharpoonup^* d\nu = S^{N/p} \delta_{x_0}, \qquad x_0 \in \overline{\Omega}, \tag{7}$$

where S is the best Sobolev constant for the critical embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and δ_{x_0} is a Dirac impulse concentrated at x_0. It is also proved that $x_0 \in \operatorname{int} \Omega$ if $\partial\Omega$ is smooth (see [9,

p. 2120]). Convergence (7) means that for any $\varphi \in C_0(\Omega)$ (the space of continuous functions with compact support in Ω), we have

$$\int_\Omega u(q_k)\varphi \longrightarrow \int_\Omega \varphi \, d\nu = S^{N/p}\varphi(x_0). \tag{8}$$

Assume now that $\Omega = \Omega_r$ and all $u(q_k)$ are radial. Choose the positive functions $\varphi_i \in C_0(\Omega_r), i = 1, 2$ as follows: $\varphi_1(x_0) = 1, \sup \varphi_1 \subset \{x \in \Omega_r: |x - x_0| < \operatorname{dist}(x_0, \partial\Omega_r)/2\}, \varphi_2(x) = \varphi_1(-x)$ for $x \in \Omega_r$. Then the radial symmetry of $u(q_k)$ implies

$$\int_{\Omega_r} u(q_k)\varphi_1 = \int_{\Omega_r} u(q_k)\varphi_2,$$

for any k. On the other hand, since $\varphi_2(x_0) = 0$, we have for k large enough,

$$\int_{\Omega_r} u(q_k)\varphi_1 > \frac{2}{3}\, S^{N/p} \qquad \text{and} \qquad \int_{\Omega_r} u(q_k)\varphi_2 < \frac{1}{3}\, S^{N/p},$$

which is absurd. Hence, there is q_k such that $u(q_k)$ is not radial which completes the proof of the theorem. The proof of the corollary follows from the fact that the equality in (2) does not depend on radial transformations when $\Omega = \Omega_r$.

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